**Random process.** Consider a random experiment specified by the outcomes $\zeta$ from some sample space $S$, the events defined on $S$, and the probabilities on $S$. Suppose that to every outcome $\zeta \in S$, we assign a function of time according to some rule $X(t,\zeta), t \in I$. The graph of $X(t,\zeta)$ vs. $t$, for $t$ fixed, is a **realization** of the RP. For each fixed $t_k$ from the index set $I$, $X(t_k,\zeta)$ is a RV. RP is an indexed family of RVs.

**Mean**. $m_X(t) = E[X(t)] = \int_{\mathbb{R}} x f_{X(t)}(x)dx$, for $f_{X(t)}$ the pdf of $X(t)$.

**Autocorrelation**. $R_X(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{\mathbb{R}}\int_{\mathbb{R}} xy f_{X(t_1)} f_{X(t_2)}(x,y)dxdy$. **Cross-correlation**: $R_{X,Y}(t_1,t_2) = E[X(t_1)Y(t_2)]$; $X,Y$ **orthogonal** if $R_{X,Y}(t_1,t_2) = 0 \;\forall\; t_1,t_2$.

**Autocovariance**. $C_X(t_1,t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}] = R_X(t_1,t_2) - m_X(t_1)m_X(t_2)$; $\mathrm{var}[X(t)] = C_X(t,t)$. **Cross-covariance**: $C_{X,Y}(t_1,t_2) = E[\{X(_t1) - m_X(t_!)\}\{Y(t_2) - m_Y(t_2)\}] = R_{X,Y}(t_1,t_2) - m_X(t_1)m_Y(t_2)$; $X,Y$ **uncorrelated** if $C_{X,Y}(t_1,t_2) = 0 \;\forall\; t_1,t_2$.

$X,Y$ **independent** if vector RVs $(X(t_1),...,X(t_k))$ and $(Y(t'_1),...,Y(t'_j))$ independent $\forall\; k,j$, choices of $t_1,...,t_k$ and $t'_1,...,t'_j$. If $X,Y$ Gaussian, $X,Y$ independent $\Leftrightarrow X,Y$ uncorrelated.

$X(t)$ **stationary** if $F_{X(t_1),...,X(t_k)}(x_1,...,x_k) = F_{X(t_1+\tau),...,X(t_k+\tau)}(x_1,...,x_k)$ $\forall\; \tau, k, \{t_1,...,t_k\}$. Also, $m_X(t) = E[X(t)] = m \;\forall\; t$; $\mathrm{var}[X(t)] = E[(X(t)-m)^2] = \sigma^2 \;\forall\; t$; $R_X(t_1,t_2) = R_X(t_2-t_1) \;\forall\; t_1,t_2$; $C_X(t_1,t_2) = C_X(t_2-t_1) \;\forall\; t_1,t_2$.

$X(t)$ **wide-sense stationary** (WSS) if $m_X(t) = m \;\forall\; t$ and $R_X(t_1,t_2) = R_X(t_2-t_1) \;\forall\; t_1,t_2$. **Average power** (second moment): $E[X^2(t)] = R_X(0) \;\forall\; t$. Also $R_X(\tau) = R_X(-\tau)$; $P[|X(t+\tau) - X(t)| > \epsilon] = \frac{2}{\epsilon^2}[R_X(0) - R_X(\tau)]$; $\lim_{\tau\to\infty} R_X(\tau) = E[X(t)]^2$; and $|R_X(\tau)| \le R_X(0)$; $R_X(0) = R_X(d) \Rightarrow R_X(\tau)$ is periodic with period $d$ and $X(t)$ is **mean square periodic**: $E[X(t+d) - X(t))^2] = 0$. $X(t)$ Gaussian, WSS $\Rightarrow X(t)$ stationary. $X(t)$ stationary $\Rightarrow X(t)$ WSS.

$X(t)$ **mean-square continuous** at $t_0$ if $E[(X(t) - X(t_0))^2] \to 0$ as $t \to 0$; i.e., $\mathrm{l.i.m.}_{t\to t_0} X(t) = X(t_0)$; if mean-square limit exists, **mean-square derivative** is $X'(t) = \frac{dX(t)}{dt} = \lim_{\epsilon\to 0} \frac{1}{\epsilon}[X(t+\epsilon,\zeta) - X(t,\zeta)]$. $X'(t)$ exists at $t$ if $\frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1,t_2)$ exists at $(t_1,t_2) = (t,t)$; for $X(t)$ WSS, $X'(t)$ exists if $\frac{d^2}{d\tau^2} R_X(\tau)$ exists at $\tau = 0$. For $X(t)$ Gaussian, $X'(t)$ also Gaussian, if it exists. $E[X'(t)] = \frac{d}{dt} m_X(t)$ ($= 0$ if $X(t)$ WSS). $R_{X'}(t_1,t_2) = \frac{\partial^2}{\partial t_1 \partial t_2} R_X(t_1,t_2)$ ($= -\frac{d^2}{d\tau^2} R_X(\tau)$ if $X(t)$ WSS). **Mean-square integral** is $Y(t) = \int_{t_0}^t X(t')dt' = \mathrm{l.i.m.}_{\Delta_k\to 0} \sum_k X(t_k)\Delta_k$. $m_Y(t) = \int_{t_0}^t m_X(t')dt'$ and $R_Y(t_1,t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} R_X(u,v)dudv$, and $Y(t)$ exists if $R_Y(t_1,t_2)$ exists.

**Power spectral density**. For $X(t)$ WSS, $S_X(f) = \mathscr{F}\{R_X(\tau)\} = \int_{\mathbb{R}} R_X(\tau) e^{-j2\pi f\tau} d\tau$. For $X(t) \in \mathbb{R}$, $R_X(\tau) = R_X(-\tau)$, so $S_X(f) = \int_{\mathbb{R}} R_X(\tau)\cos 2\pi f\tau d\tau = S_X(-f) \ge 0$. $R_X(\tau) = \mathscr{F}^{-1}\{S_X(f)\} = \int_{\mathbb{R}} S_X(f) e^{j2\pi f\tau} df$. **Average power** of $X(t)$: $E[X^2(t)] = R_X(0) = \int_{\mathbb{R}} S_X(f)df$. **Cross-power spectral density**: $S_{X,Y}(f) = \mathscr{F}\{R_{X,Y}(\tau)\}$, where $R_{X,Y}(\tau) = E[X(t+\tau)Y(t)]$. For $X(t)$ input into a linear system with impulse response $h(t)$ and output $Y(t)$, $S_Y(f) = |H(f)|^2 S_X(f)$.

$X(t)$ **Markov** if for arbitrary times $t_1 < t_2 < ... < t_k < t_{k+1}$, $P[X(t_{k+1}) = x_{k+1}|X(t_k) = x_k,...,X(t_1) = x_1] = P[X(t_{k+1} = x_{k+1}|X(t_k) = x_k]$. $X(t)$ is the **state** of the process at time $t$. **Markov chain** is an integer-valued Markov process. If $P[X_{n+1} = j|X_n = i] = p_{ij} \;\forall\; n$, $X_n$ has **homogeneous transition probabilities** and $P[X_n = i_n,...,X_0 = i_0] = p_{i_{n-1},i_n} \dots p_{i_0,i_1} p_{i_0}(0)$. $X_n$ is completely specified by the initial pmf $p_i(0)$ and the **transition probability matrix** $P = p_{ij}, i,j = 0,1,....$ Note $1 = \sum_j P[X_{n+1} = j|X_n = i] = \sum_j p_{ij}$. $n$-step transition probability matrix: $P(n) = P^n$. State pmf at time $n$: $\boldsymbol{p}(n) = \boldsymbol{p}(0)P^n, n = 1,2,...$ . $X_n$ **irreducible** if $\forall\; i,j \;\exists\; n \ge 0$ st $P[X_n = j|X_0 = i] > 0$. State $i$ has **period** $k = \gcd\{n \text{ st } p_{ii}(n) > 0\}$. Even if a state $i$ has period $k$, it may not be possible to return to $i$ in $k$ steps. Irreducible $X_n$ **aperiodic** if all states in its single class have period 1. **Stationary state pmf**: $\pi_j = \sum_i p_{ij}\pi_i$ st $\sum_i \pi_i = 1$. MC is stationary if $\boldsymbol{p}(0) = \boldsymbol{\pi}$, as $\boldsymbol{p}(n) = \boldsymbol{\pi}P^n = \boldsymbol{\pi} \;\forall\; n$.

Consider a **system** in which an input signal $x(t)$ is mapped to an output signal $y(t)$ by the transformation $y(t) = T[x(t)]$. System is **linear** if $T[\alpha x_1(t) + \beta x_2(t)] = \alpha T[x_1(t)] + \beta T[x_2(t)]$. It is **time-invariant** if the response to $x(t-\tau)$ is $y(t-\tau)$. **Impulse response** of an LTI system is $h(t) = T[\delta(t)]$ for $\delta(t)$ a unit delta function applied at $t = 0$. For arbitrary input $x(t)$, $y(t) = h(t) * x(t) = \int_{\mathbb{R}} h(s)x(t-s)ds = \int_{\mathbb{R}} h(t-s)x(s)ds$. **Transfer function** $H(f) = \mathscr{F}\{h(t)\} = \int_{\mathbb{R}} h(t)e^{-j2\pi ft}dt$.

For $X_t, Y_t$ disc.-time, zero-mean, jointly WSS and $Y_t = \sum_{\beta=t-\alpha}^{t+\beta} h_{t-\beta}X_\beta = \sum_{\beta=-b}^{a} h_\beta X_{t-\beta}$, the **optimum filter** for estimating $Z_t$ satisfies $R_{Z,X}(m) = \sum_{\beta=-b}^{a} h_\beta R_X(m-\beta)$ for $-b \le m \le a$ and has $E[(Z_t - Y_t)^2] = R_Z(0) - \sum_{\beta=-b}^{a} h_\beta R_{Z,X}(\beta)$. If we wish to predict $Z_n$ in terms of $Z_{n-1}, Z_{n-2},...,Z_{n-p}$, we have a **linear prediction** problem: $X_\alpha = Z_\alpha$ and $R_Z(m) = \sum_{\beta=1}^{p} h_\beta R_Z(m-\beta)$ for $m \in \{1,...,p\}$, or $[R_Z(1), R_Z(2),...,R_Z(p)]^T = \boldsymbol{R_Z}[h_1,h_2,...,h_p]^T$, where

$$\boldsymbol{R_Z} = \begin{bmatrix} R_Z(0) & R_Z(1) & R_Z(2) & \dots & R_Z(p-1) \\ R_Z(1) & R_Z(0) & R_Z(1) & \dots & R_Z(p-2) \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & R_Z(1) \\ R_Z(p-1) & \cdot & \cdot & R_Z(1) & R_Z(0) \end{bmatrix}$$

and $E[e_n^2] = R_Z(0) - \sum_{\beta=1}^{p} h_\beta R_Z(\beta)$.

$X(t)$ has **independent increments** if for any $k$ and any choice of sampling instants $t_1 < t_2 < ... < t_k$, the RVs $X(t_2)-X(t_1), X(t_3)-X(t_2), ..., X(t_k)-X(t_{k-1})$ are independent RVs. $X(t)$ has indep. increments $\Rightarrow$ $X(t)$ Markov. If increments of the same length have the same distribution regardless of time origin, i.e., if $P[X_{n'} - X_n = y] = P[X_{n'-n} = y]$, $X_n$ has **stationary increments**.

$X(t)$ **Gaussian** if the samples $X_1 = X(t_1), X_2 = X(t_2), ..., X_k = X(t_k)$ are jointly Gaussian RVs $\forall\ k, t_1, ..., t_k$. Joint pdf: $f_{X_1,...,X_k}(x_1, ..., x_k) = \exp\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{m})^T K^{-1}(\boldsymbol{x}-\boldsymbol{m})\}(2\pi)^{-\frac{k}{2}}|K|^{-\frac{1}{2}}$, where $\boldsymbol{m} = [m_X(t_1), ..., m_X(t_k)]^T$, $K_{ij} = C_X(t_i, t_j)$. The sum of Gaussian processes is Gaussian, with joint pdf $f_{S_{n_1},S_{n_2}}(y_1, y_2) = f_{S_{n_2-n_1}}(y_2 - y_1) f_{S_{n_1}}(y_1)$.

$X_n$ **iid** if it consists of a sequence of iid RVs with common cdf $F_X(x)$, mean $m$, variance $\sigma^2$. Joint cdf: $F_{X_1,...,X_k}(x_1, ..., x_k) = F_X(x_1)F_X(x_2)...F_X(x_k)$. Mean: $m_X(n) = m\ \forall\ n$. Autocovariance: $C_X(n_1, n_2) = \sigma^2 \delta_{n_1,n_2}$ for $\delta_{ij}$ the Kronecker delta. Autocorrelation: $R_X(n_1, n_2) = C_X(n_1, n_2) + m^2$. iid process is Markov.

**Bernoulli process** $I_n$ is a sequence of independent Bernoulli RVs with mean $p$. $m_I(n) = p$, $\text{var}[I_n] = p(1-p)$. Bernoulli process is iid.

**Sum process** $S_n = X_1 + X_2 + ... + X_n = S_{n-1} + X_n$ where $S_n = 0$. $S_n$ is Markov and ISI: $P[S_{n_1} = y_1, S_{n_2} = y_2, ..., S_{n_k} = y_k] = P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1] \dots P[S_{n_k-n_{k-1}} = y_k - y_{k-1}]$. $m_S(n) = E[S_n] = nE[X] = nm$. $\text{var}[S_n] = n\text{var}[X] = n\sigma^2$. $C_S(n, k) = \min(n, k)\sigma^2$.

**Random step**. Let $D_n = 2I_n - 1$, where $I_n$ is the Bernoulli process. Then $D_n = 1$ if $I_n = 1$ and $D_n = -1$ if $I_n = 0$. $m_D(n) = 2p - 1$ and $\text{var}[D_n] = 4p(1-p)$. $D_n$ is iid.

**Binomial counting process**. Let the $I_i$ be a Bernoulli process with mean $p$, and let $S_n$ be the corresponding sum process. $S_n$ is the counting process that gives the number of successes in the first $n$ Bernoulli trials, and is a binomial RV with parameters $n$ and $p$, $P[S_n = j] = {}_nC_j p^j (1-p)^{n-j}$, for $0 \le j \le n$, and 0 otherwise. $m_S(n) = np$, $\text{var}[S_n] = np(1-p)$. $S_n$ is ISI. Joint pmf at times $n_1, n_2$: $P[S_{n_1} = y_1, S_{n_2} = y_2] = P[S_{n_1} = y_1]P[S_{n_2-n_1} = y_2 - y_1] = {}_{n_2-n_1}C_{y_2-y_1}\,{}_{n_1}C_{y_1} p^{y_2}(1-p)^{n_2-y_2}$.

**Random walk**. Let $D_n$ be the random step process and $S_n$ be the corresponding sum process. $S_n$ is a one-dimensional random walk process, with pmf $P[S_n = 2k - n] = {}_nC_k p^k (1-p)^{n-k}$ for $k \in \{0, 1, ..., n\}$. $C_S(n, k) = \min(n, k)4p(1-p)$. Random walk is ISI.

**Poisson process**. Consider a situation in which events occur at random instants of time at an average rate of $\lambda$ events per second, and $N(t)$ is the number of event occurrences in $[0, t]$. $P[N(t) = k] = \frac{(\lambda t)^k}{k!}e^{-\lambda t}$, $k = 0, 1, ...$; $N(t)$ is the Poisson process and is ISI: $P[N(t_1) = i, N(t_2) = j] = P[N(t_1) = i]P[N(t_2) - N(t_1) = j - i] = P[N(t_1) = i]P[N(t_2 - t_1) = j - i]$. $m_N(t) = \text{var}[N(t)] = \lambda t$; $C_N(t_1, t_2) = \lambda \min(t_1, t_2)$. Interevent time $T$ is an exponential RV with parameter $\lambda$: $P[T > t] = e^{-\lambda t}$; $E[T] = \frac{1}{\lambda}$, $\text{var}[T] = \frac{1}{\lambda^2}$. Interevent times form an iid sequence of exponential RVs. For $T_j$ the iid exponential interarrival times, if $S_n = \sum_{j=1}^n T_j$ (the time at which the $n$th event occurs in the Poisson process), then $f_{S_n}(y) = \frac{(\lambda y)^{n-1}}{(n-1)!}\lambda e^{-\lambda y}$ for $y \ge 0$ (Erlang RV). Poisson process is Markov. Arrivals "at random": given only one arrival occurred in $[0, t]$, at time $X$, let $N(x)$ be the number of events up to time $x$ $(0 < x < t)$ and let $N(t) - N(x)$ be the increment in the interval $(x, t]$. Then $P[X \le x] = P[N(x) = 1|N(t) = 1] = \frac{x}{t}$; the arrival time of the given arrival is uniformly distributed in $[0, t]$.

**Random telegraph signal**. $X(t)$ assumes values $\{-1, 1\}$, and $X(0)$ is either with probability $\frac{1}{2}$. $X(t)$ changes polarity with each occurrence of an event in a Poisson process with rate $\alpha$. $E[X(t)] = 0$, $\text{var}[X(t)] = 1$, $C_X(t_1, t_2) = \exp\{-2\alpha|t_2 - t_1|\}$. Is Markov.

**Wiener process**. $X(t)$ is a cont.-time symmetric random walk. $E[X(t)] = 0$, $\text{var}[X(t)] = \alpha t$, $C_X(t_1, t_2) = \alpha \min(t_1, t_2)$. Is Gaussian, Markov. Is the mean-square integral of white Gaussian noise process.

**White Gaussian noise**. $R_X(t_1, t_2) = \alpha\delta(t_1 - t_2)$. Is the mean square derivative of the Wiener process.

**Fourier transform.** For $G(f) = \mathscr{F}\{g(t)\} = \int_{\mathbb{R}} g(t)e^{-j2\pi ft}dt$, $g(t) = \mathscr{F}^{-1}\{G(f)\} = \int_{\mathbb{R}} G(f)e^{j2\pi ft}dt$. Linearity: $\mathscr{F}\{ag_1(t) + bg_2(t)\} = aG_1(f) + bG_2(f)$. Time scaling: $\mathscr{F}\{g(at)\} = G(f/a)/|a|$. Duality: If $\mathscr{F}\{g(t)\} = G(f)$, then $\mathscr{F}\{G(t)\} = g(-f)$. Time shifting: $\mathscr{F}\{g(t-t_0)\} = G(f)e^{-j2\pi ft_0}$. Frequency shifting: $\mathscr{F}\{g(t)e^{j2\pi f_0 t}\} = G(f - f_0)$. Differentiation: $\mathscr{F}\{g'(t)\} = j2\pi fG(f)$. Integration: $\mathscr{F}\{\int_{-\infty}^t g(s)ds\} = \frac{G(f)}{j2\pi f} + \frac{1}{2}G(0)\delta(f)$. Multiplication in time: $\mathscr{F}\{g_1(t)g_2(t)\} = G_1(f) * G_2(f)$. Convolution in time: $\mathscr{F}\{g_1(t) * g_2(t)\} = G_1(f)G_2(f)$.

**Fourier transform pairs.**

| $g(t)$ | $G(f)$ |
|---|---|
| $1$, $t \in [-T, T]$; 0 otherwise | $\frac{2T}{2\pi fT}\sin 2\pi fT$ |
| $\frac{2W}{2\pi Wt}\sin 2\pi Wt$ | $1$, $t \in [-W, W]$; 0 otherwise |
| $1 + \frac{1}{T}$, $t \in [-T, 1]$; $1 - \frac{1}{T}$, $t \in [1, T]$ | $T\left(\frac{\sin \pi fT}{\pi fT}\right)^2$ |
| $\delta(t)$ | $1$ |
| $1$ | $\delta(t)$ |